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On Kinks in the Gross-Neveu Model

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Abstract

We investigate static space dependent $\sigma(x) = \langle\bar{\psi}\psi\rangle_{vac}$ saddle point configurations in the two dimensional Gross-Neveu model in the large N limit. We analyse the saddle point condition for $\sigma(x)$ employing supersymmetric quantum mechanics and using simple properties of the diagonal resolvent of one dimensional Schrödinger operators. We concentrate on the sector of unbroken supersymmetry. We solve the saddle point equation in this sector explicitly. The resulting solutions are the Callan-Coleman-Gross-Zee kink configurations. We thus provide a direct and clean construction of these kinks. Our method of finding such non-trivial static configurations may be applied also in other two dimensional field theories.

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I Introduction

The Gross-Neveu model [1] is a well known two dimensional field theory of N massless Dirac fermions $\psi_a\,(a = 1,\ldots,N)$ with $\mathcal{U}(N)$ invariant self interactions, whose action is given by

$$S = \int d^2x \left[\sum_{a=1}^{N} \bar{\psi}_a\, i\partial\!\!\!/\, \psi_a + \frac{g^2}{2}\left(\sum_{a=1}^{N} \bar{\psi}_a\,\psi_a\right)^2\right] . \tag{1}$$

We are interested in the large N limit of (1) in which $N \to \infty$ while Ng^2 is held fixed. Decomposing each Dirac spinor into two Majorana spinors one observes that S is invariant under $\mathcal{O}(2N)$ flavour symmetry containing the $\mathcal{U}(N)$ mentioned above as a subgroup [2]. The field theory defined by (1) is a renormalisable field theory exhibiting asymptotic freedom, dynamical symmetry breaking and dimensional transmutation. Its spectrum was calculated semiclassically (in the large N limit) in [2]. It contains the fermions in (1) which become massive, as well as a rich collection of bound states thereof. The spectrum of (1) contains also kink configurations [3]. We refer to these as the Callan-Coleman-Gross-Zee (CCGZ) kinks in the sequel. These kinks are expected to be part of the spectrum of the Gross-Neveu model since dynamical breaking of the discrete chiral symmetry in the Gross-Neveu model suggests that there should be extremal field configurations that interpolate between the two minima of the *effective* potential associated with (1) in much the same way that such configurations arise in classical field theories whose potential term has two or more equivalent minima.

The appearance of such solitons in this model implies the existence of an infinite number of conservation laws forbidding particle production in scattering processes and enables the exact calculation of S-matrix elements in the various sectors of the model [4].

In this paper we discuss static space dependent $\sigma(x) = \langle\bar{\psi}\psi\rangle_{vac}$ configurations that are solutions of the saddle point equation governing the effective action corresponding to (1) as $N \to \infty$. Such $\sigma(x)$ configurations correspond to non-trivial excitations of the vacuum [5, 6] and are therefore important in determining the entire spectrum of the field theory in question[2] and its finite temperature behaviour as well[7]. Such configurations are important also in discussing the behaviour of the $\frac{1}{N}$ expansion of (1) at large orders [8, 9, 10].

Our discussion makes use of supersymmetric quantum mechanics and simple properties of the diagonal resolvent of one dimensional Schrödinger operators. This suppersymmetry relates the upper and lower components of spinors, implying that the square of the Dirac operator may be decomposed into two isospectral Schrödinger operators. It is closely related to the soliton degeneracy discussed in [11], where the soliton is considered as a degenerate doublet having fermion numbers $\pm\frac{1}{2}$. Concentrating on the sector of unbroken supersymmetry we are able to solve the saddle point equation explicitly. These explicit solutions are the CCGZ kink configurations. Our explicit solution of the saddle point equation governing the large N behaviour of (1) provides a clean direct construction of these kinks.

In a recent paper [12], we have applied a similar method to the anharmonic $\mathcal{O}(N)$ oscillator and the two dimensional $\mathcal{O}(N)$ vector model in the limit $N \to \infty$. In the latter case we have found that the effective action sustains *approximate* extremal bilinear condensates of the $\mathcal{O}(N)$ vector field that vary very slowly in time. The static part of these configurations turned out to be analogous to the CCGZ kinks in the Gross-Neveu model.

The paper is organised as follows: In section II we analyse the saddle point equation for static $\sigma(x)$ configurations using supersymmetric quantum mechanics. We resolve the saddle point equation into frequencies and demand that the extremum condition be satisfied by each Fourier component separately. This strong condition turns out to leave us always in the sector of unbroken supersymmetry. This corresponds physically to $\sigma(x)$ configurations that interpolate between the two vacua of the Gross-Neveu model at the two ends of the world, which are the CCGZ kinks that we find as explicit solutions of the saddle point equation in section III. We conclude our discussion in section IV.

II The Saddle Point Equation for Static Solutions

Following[1] we rewrite (1) as

$$S = \int d^2x \left[i\bar{\psi}\partial\!\!\!/\psi - g\sigma\bar{\psi}\psi - \frac{1}{2}\sigma^2 \right] \tag{2}$$

where $\sigma(x)$ is an auxiliary field [13].

Thus, the partition function associated with (2) is

$$\mathcal{Z} = \int \mathcal{D}\sigma\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \exp\left\{ i\int\limits_{-\infty}^{\infty} dt \int\limits_{-\infty}^{\infty} dx \left[\bar{\psi} i\partial\!\!\!/\psi - g\sigma\bar{\psi}\psi - \frac{1}{2}\sigma^2 \right] \right\} \tag{3}$$

Gaussian integration over the grassmannian variables is straightforward, leading to $\mathcal{Z} = \int \mathcal{D}\sigma\, \exp\{iS_{eff}[\sigma]\}$ where the bare effective action is [14]

$$S_{eff}[\sigma] = -\frac{1}{2}\int\limits_{-\infty}^{\infty} dt \int\limits_{-\infty}^{\infty} dx\sigma^2 - i\frac{N}{2}\,\mathrm{Tr}\ln\left[-(i\partial\!\!\!/ - g\sigma)(i\partial\!\!\!/ + g\sigma)\right]\,. \tag{4}$$

The ground state of the Gross-Neveu model (1) is described by the simplest extremum of S_{eff}[1] in which $\sigma = \sigma_0$ is a constant that is fixed by the (bare) gap equation

$$-\,g\sigma_0 + iNg^2\,\mathrm{tr}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k\!\!\!/ - g\sigma_0} = 0\,. \tag{5}$$

Therefore, the dynamically generated mass of small fluctuations of the Dirac fields around this vacuum is

$$m = g\sigma_0 = \mu\, e^{1-\frac{\pi}{Ng_R^2(\mu)}} \tag{6}$$

where μ is an arbitrary renormalisation scale, and the renormalised coupling $g_R\,(\mu)$ is related to the cut-off dependent bare coupling via $\Lambda\, e^{-\frac{\pi}{Ng^2(\Lambda)}} = \mu\, e^{1-\frac{\pi}{Ng_R^2(\mu)}}$ where Λ is an ultraviolet cutoff. Since m is the physical mass of the fermions it must be a renormalisation group invariant, and this fixes the scale dependence of the renormalised coupling g_R ,namely, equation (6). From now on we will drop the subscript R from the renormalised coupling and simply denote it by Ng^2.

As was explained in the introduction, we are interested in more complicated extrema of S_{eff}, namely static space dependent solutions of the extremum condition on

S_{eff}. This condition reads generally

$$\frac{\delta S_{eff}}{\delta\sigma(x,t)} = -\sigma(x,t)$$
$$-i\frac{N}{2}\mathrm{tr}\left\{\left[2g^2\sigma(x,t)+ig\gamma^\mu\partial_\mu\right]\langle x,t|\,[\,\Box+g^2\sigma^2-ig\gamma^\mu\partial_\mu\sigma]^{-1}|x,t\rangle\right\}=0 \quad (7)$$

where "tr" is a trace over Dirac indices.

Specialising to static $\sigma(x)$ configurations, and using the Majorana respresentation $\gamma^1 = i\sigma_3$, $\gamma^0 = \sigma_2$ for γ matrices, (7) becomes

$$\frac{2i}{Ng}\sigma(x) = \mathrm{tr}\left[\begin{pmatrix} 2g\sigma-\partial_x & 0 \\ 0 & 2g\sigma+\partial_x \end{pmatrix}\int\limits_{-\infty}^{\infty}\frac{d\omega}{2\pi}\begin{pmatrix} R_+(x,\omega^2) & 0 \\ 0 & R_-(x,\omega^2)\end{pmatrix}\right] =$$

$$\int\limits_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[(2g\sigma-\partial_x)\,R_+\left(x,\omega^2\right)+(2g\sigma+\partial_x)\,R_-\left(x,\omega^2\right)\right] \quad (8)$$

where

$$R_\pm(x,\omega) = \langle x|\frac{1}{h_\pm-\omega^2}\,|x\rangle \quad (9)$$

are the diagonal resolvents of the one dimensional Schrödinger operators

$$h_\pm = -\partial_x^2+g^2\sigma^2\pm g\sigma'(x) \quad (10)$$

evaluated at spectral parameter ω^2.

Note that $h_\pm$ are positive semidefinite isopectral (up to zero-modes) hamiltonians, since (10) may be brought into the form [17, 18]

$$h_+ = Q^\dagger Q \;,\; h_- = QQ^\dagger \quad where$$
$$Q = -\frac{d}{dx}+g\sigma \;,\; Q^\dagger = \frac{d}{dx}+g\sigma. \quad (11)$$

These operators may be composed into a supersymmetric hamiltonian $H = \begin{pmatrix} h_+ & 0 \\ 0 & h_-\end{pmatrix}$ describing one bosonic and one fermionic degrees of freedom [17], which we identify with the upper and lower components of the spinors. Supersymmetry implies here that an interchange of the bosonic and fermionic sectors of H leaves dynamics unchanged, as can be seen from the fact that Eqs. (8)-(11) are invariant under the simultaneous interchanges

$$\begin{aligned} &\sigma \rightarrow -\sigma \quad , \quad h_{\pm} \rightarrow h_{\mp}, \\ &R_{\pm} \rightarrow R_{\mp}. \end{aligned} \tag{12}$$

Isospectrality of h_+ and h_- alluded to above means that to each eigenvector ψ_n of h_- with a *positive* eigenvalue E_n, there is a corresponding eigenvector ϕ_n of h_+ with the same eigenvalue and norm, and vice-versa. The precise form of this pairing relation is

$$\begin{aligned} \phi_n &= \frac{1}{\sqrt{E_n}} Q^{\dagger} \psi_n, \\ \psi_n &= \frac{1}{\sqrt{E_n}} Q \psi_n \quad ; \quad E_n > 0\,. \end{aligned} \tag{13}$$

It is clear that the pairing in (13) fails when $E_n = 0$. Thus, in general one cannot relate the eigenvectors with zero-eigenvalue (i.e.-the normalisable zero-modes) of one hamiltonian in (10) to these of the other. Should such a normalisable zero-mode appear in the spectrum of one of the positive semidefinite operators in (10), it must be the ground state of that hamiltonian. In this case the lowest eigenvalue of the supersymmetric hamiltonian H is zero, which is the case of unbroken supersymmetry. If such a normalisable zero-mode does not appear in the spectrum, all eigenvalues of H, and in particular-its ground state energy, are positive, and supersymmetry is broken. Since the ground state of a Schrödinger operator is non-degenerated, $h_{\pm}$ can have each no more than one such a normalisable zero-mode. Moreover, it is clear from (10) that in our one dimensional case, only *one* operator in (10) may have a normalisable zero-mode, since it must be annihilated either by Q or by $Q^{\dagger}$. In cases of unbroken supersymmetry, we will take such a normalisable zero-mode to be an eigenstate of h_-, namely, the *real* function

$$\Psi_0(x) = \mathcal{N} e^{-g \int\limits_0^x \sigma(y) dy} \tag{14}$$

which is the normalisable solution of the differential equation

$$Q^{\dagger}\Psi_0 = 0 \tag{15}$$

where $\mathcal{N}$ is a normalisation coefficient.

Note that a necessary condition for the normalisability of Ψ_0 is that $\sigma(x)$ have the opposite behaviour at $\pm\infty$. Thus, physically, cases of unbroken supersymmetry lead to $\sigma(x)$ configurations that interpolate between the two vacua of (4) at the two ends of the world, while cases of broken supersymmetry yield $\sigma(x)$ configurations that leave and return to the same vacuum state.

Assigning the zero-mode to h_- poses no loss of generality, since the other possible case is related to this one via (12). In what follows we will denote the right hand side of (14) by Ψ_0 also in cases of broken supersymmetry where it is non-normalisable. This should not cause any confusion, since the ground state will be denoted by ψ_0, which will be equal to Ψ_0 when supersymmetry is unbroken.

By definition, the diagonal resolvents in (9) are given by the eigenfunction expansions

$$\begin{aligned} R_-(x) &= \sum_{n=0}^{\infty} \frac{|\psi_n(x)|^2}{E_n - \omega^2}\,, \\ R_+(x) &= \sum_{n>0}^{\infty} \frac{|\phi_n(x)|^2}{E_n - \omega^2} \end{aligned} \tag{16}$$

where the sums extend over all eigenstates, including the continua of scattering states where they are understood as integrals.

Using (13), R_+ may be expressed in terms of the ψ_n's as

$$R_+(x,\omega^2) = \sum_{n>0}^{\infty} \frac{1}{E_n\Psi_0^2(x)} \frac{|W_n(x)|^2}{E_n - \omega^2} \tag{17}$$

where $W_n(x) = \Psi_0(x)\psi_n'(x) - \Psi_0'(x)\psi_n(x)$ is the wronskian of Ψ_0 and ψ_n. An elementary consequence of the Schrödinger equation is that $W_n'(x) = -E_n\Psi_0(x)\psi_n(x)$. Using this relation, (16) and (17) imply the important relation

$$(2g\sigma - \partial_x)\,R_+ + (2g\sigma + \partial_x)\,R_- = 2\left(2g\sigma + \frac{d}{dx}\right)\langle x|\,\mathcal{P}\frac{1}{h_- - \omega^2}\,|x\rangle\,. \tag{18}$$

Here $\mathcal{P}$ is the projector

$$\mathcal{P} = 1 - \lambda|\psi_0\rangle\langle\psi_0| \tag{19}$$

that projects out the ground state of h_- when supersymmetry is unbroken ($\lambda = 1$), and is just the unit operator otherwise ($\lambda = 0$). We can also use (5) to make a frequency resolution of unity as

$$\frac{i}{Ng^2} = \int_{-\Lambda}^{\Lambda} \frac{d\omega}{\pi} \langle x|\frac{1}{-\partial_x^2 + m^2 - \omega^2 - i\epsilon}|x\rangle \,. \tag{20}$$

Moreover, from (18), (20) and the elementary relation

$$\langle x|\frac{1}{-\partial_x^2 + m^2 - \omega^2 - i\epsilon}|x\rangle = \frac{1}{2\sqrt{m^2 - \omega^2 - i\epsilon}}\,, \tag{21}$$

the frequency resolution of (8) becomes

$$\frac{d}{dx}\langle x|\mathcal{P}\frac{1}{h_- - \omega^2}|x\rangle = 2g\sigma(x)\left[\frac{1}{2\sqrt{m^2 - \omega^2}} - \langle x|\mathcal{P}\frac{1}{h_- - \omega^2}|x\rangle\right] .$$

Substituting (19) into the last equation, all dependence on λ cancels out and the extremum condition (8) is shaped into its final form as the differential condition on R_-

$$\frac{dR_-(x,\omega^2)}{dx} = 2g\sigma(x)\left[\frac{1}{2\sqrt{m^2 - \omega^2}} - R_-\left(x,\omega^2\right)\right] \tag{22}$$

which has to be satisfied regardless of whether supersymmetry is broken or not.

Now, $R_-(x,\omega^2)$, being the diagonal resolvent of h_- at spectral parameter ω^2 is subjected to the so-called "Gelfand-Dikii" equation[15][16]

$$-2R_-(x,\omega^2)R_-''(x,\omega^2) + \left(R_-'(x,\omega^2)\right)^2 + 4R_-^2(x,\omega^2)\left[g^2\sigma^2 - g\sigma' - \omega^2\right] = 1. \tag{23}$$

Therefore, both equations (22) and (23) must hold and they form a system of coupled non-linear differential equations in the unknowns $\sigma(x)$, and $R_-(x,\omega^2)$. Substituting R_-' and R_-'' from (22) into (23) we obtain a quadratic equation for R_- whose solutions are

$$R_-\left(x,\omega^2\right) = \frac{-g\sigma' \pm \sqrt{(g\sigma')^2 + 4\omega^2 g^2\sigma^2 - 4\omega^2\left(m^2 - \omega^2\right)}}{4\omega^2\sqrt{m^2 - \omega^2}}\,. \tag{24}$$

To see what the two signs of the square root correspond to we observe that the solution with the negative sign in front of the square root has a simple pole as a function of ω^2 at $\omega^2 = 0$ with a negative residue, while the other solution is regular and positive at $\omega^2 = 0$. Therefore, from (16) it is clear that the negative sign root corresponds to the case of unbroken supersymmetry, where the simple pole signals the existence of a normalisable zero-mode in the spectrum of h_-, while the positive root solution corresponds, for similar reasons, to cases in which h_- lacks such a zero-mode. We will see in the next section that the latter solution corresponds also to the case of unbroken supersymmetry, where the zero-mode is in the spectrum of h_+. Note also the branch point singularity in (24) at $\omega^2 = m^2$ which signals the threshold of the continuous part of the spectrum of h_-.

Finally, we observe that in principle, a substitution of (24) into (22) yields a second order non-linear equation for $\sigma(x)$. However, doing so is not really necessary, and all desired information could be deduced already from (24) itself as we show in the next section.

III Static Solutions to the Extremeum Condition

Considering the explicit form for $R_-(x,\omega^2)$ in (24) we have seen at the end of the previous section that $R_-(x,\omega^2)$ has generally a branch point singularity at $\omega^2 = m^2$ which is the threshold of the continuum part of the spectrum of h_-. Clearly, h_- can have no continuous spectrum other than that starting at $\omega^2 = m^2$, thus leading to the conclusion that R_- in (24) can have no other branch points in the ω^2 plane. Therefore, the expression under the square root in (24) must be a perfect square as a polynomial in ω^2, namely,

$$\pm\, g\sigma'(x) = g^2\sigma^2(x) - m^2\,. \tag{25}$$

From (25) we find straight forwardly the solutions

$$g\sigma(x) = \pm\, m\, tanh\, [m\,(x - x_0)] \tag{26}$$

which are exactly the CCGZ kinks and anti-kinks. Here the parameter x_0 is an integration constant that implies translational invariance of (26) since it is the arbitrary location of the kinks. Clearly, both cases in (26), and therefore both cases in (24), lead to $h_\pm$ operators that do not break supersymmetry, since (14),(26) imply that

$$\Psi_0 = \left(\frac{m}{2}\right)^{\frac{1}{2}} e^{-g\int_0^x \sigma(y)dy} = \left(\frac{m}{2}\right)^{\frac{1}{2}} sech[m(x - x_0)] \tag{27}$$

is the normalisable zero-mode of h_- for the kink configuration, and of h_+ when $\sigma(x)$ is the anti-kink. Thus, we come to the important conclusion that the *only* possible static extremal $\sigma(x)$ configurations in the sector of unbroken supersymmetry are CCGZ kinks and anti-kinks. As it stands, our frequency decomposition of the extremum condition (22) seems to lead always to extremal $\sigma(x)$ configurations that do not break supersymmetry. We are thus unable to find in this manner the extremal $\sigma(x)$ configurations found in [2] in which $\sigma(x)$ has the shape of kink anti-kink pair that are very close to each other. Such a configuration evidently breaks supersymmetry [17].

When $\sigma(x)$ is a kink (24) becomes

$$R_-(x,\omega^2) = -\frac{m^2\ sech^2\,[m\,(x-x_0)]}{2\omega^2\sqrt{m^2-\omega^2}} + \frac{1}{\sqrt{m^2-\omega^2}} \tag{28}$$

while for anti-kinks it is just the expression on the right hand side of (21).

These statements on R_- are consistent with the explicit form of the hamiltonians $h_\pm$. Using (10),(26) we find in the kink case

$$\begin{aligned} h_+ &= -\partial_x^2 + g^2\sigma^2 + g\sigma' = -\partial_x^2 + m^2 \quad \text{and} \\ h_- &= -\partial_x^2 + g^2\sigma^2 - g\sigma' = -\partial_x^2 + m^2 - 2m^2\,sech^2[m(x-x_0)] \end{aligned} \tag{29}$$

while in the anti-kink case $h_\pm$ interchange their roles . Thus, in the latter case, h_- becomes the Schrödinger operator of a freely moving particle in a *constant* potential m^2 which is the reason why R_- is given by the simple expression in (21) in the anti-kink case. Indeed, that expression is the x independent solution of the Gelfand-Dikii equation (23) corresponding to the constant potential $g^2\sigma^2 - g\sigma' = m^2$ that is positive at $\omega^2 = 0$.

In the kink case, we have obtained the explicit form (28) for R_- by substituting (26) into (24). As an independent check, we may deduce this expression for the diagonal resolvent for the potential $g^2\sigma^2 - g\sigma' = m^2 - 2m^2\,sech[m(x-x_0)]$ in other ways. The simplest one is to apply an ansatz of the form $R_- = \alpha\,sech^2(\beta x) + \gamma$ to the Gelfand-Dikii equation. Another way to derive it is to use the general formula $R(x) = G(x,y)|_{x=y} = \frac{\psi_1(x_>)\psi_2(x_<)}{W(\psi_1,\psi_2)}|_{x=y}$ where the ψ's are the so called Jost functions of the problem. In this case they are hypergeometric functions multiplied by *sech* factors. The wronskian in this expression is a ratio of Γ functions dependent on ω^2 and m^2[6].

We can deduce from (24) the CCGZ solution (26) to the extremum condition (22),(23) by yet another method which does not rely upon the brach-cut structure of R_- but rather on its pole structure. Considering the case where the zero-mode is in the spectrum of h_- we make a Laurent expansion of the appropriate expression for

R_- in (24) around the simple pole at $\omega^2 = 0$. The leading term in this expansion is

$$R_- \sim -\frac{1}{\omega^2}\left[\frac{g\sigma'}{2m} + \mathcal{O}\left(\omega^2\right)\right]. \tag{30}$$

Comparing (30) to (14) and (16) we find

$$\Psi_0^2(x) = \mathcal{N}^2 exp\left(-2g\int\limits_0^x \sigma(y)dy\right) = \frac{g\sigma'}{2m} \tag{31}$$

which yields

$$\mathcal{N}^2 e^{-2\phi} = \frac{\phi''}{2m} \quad ; \quad \phi(x) = g\int\limits_0^x \sigma(y)dy\,. \tag{32}$$

Solving (32) we find

$$\phi(x) = ln\, cosh[m(x - x_0)] + c \tag{33}$$

where c and x_0 are integration constants and we have imposed the normalisation condition $\int\limits_{-\infty}^{\infty} \Psi_0^2\, dx = 1$ to fix $\mathcal{N} = \sqrt{\frac{m}{2c}}$. Clearly, then, (33) yields the CCGZ kinks upon differentiation.

We are now in a position to verify briefly that the CCGZ kink configuration leads an h_- operator that has indeed a single normalisable zero-mode as the explicit form (28) for R_- suggests. Our discussion follows [18]. In the kink sector (29) implies that the eigenstates of h_+ are simply these of freely moving particles $\phi_k(x) = e^{ikx}$, with a *continuum* of strictly positive eigenvalues $E_+ = k^2 + m^2 \geq m^2$. These states are isospectral to the eigenstates of h_-,

$$\psi_k(x) = \frac{1}{\sqrt{k^2+m^2}} Q\,\phi_k(x) = \left\{\frac{m\, tanh\,[m(x - x_0)] - ik}{\sqrt{k^2+m^2}}\right\} e^{ikx} \tag{34}$$

which are therefore the *scattering* states of h_-. The S matrix associated with h_- is thus

$$S(k) = \frac{ik - m}{ik + m} = exp\left[i\left(\pi - 2\; arctan\frac{k}{m}\right)\right]. \tag{35}$$

Since h_+ in (29) has only scattering states, h_- can have no bound states other than its zero-mode (27) which must therefore be its ground state. This *single* normalisable state of h_- corresponds to the single pole of $S(k)$ in (35) at $k = im$. Note further that there are no reflected waves in any of the scattering eigenstates (34) of the Schrödinger operators in (29) . This is also the case for the supersymmetry breaking $\sigma(x)$ configurations in [2] as well as in other exactly soluble models in two space-time dimensions[2].

The fact that $h_\pm$ evaluated at the extremal point must be reflectionless can be deduced even without solving the extremum condition (22) explicitly, provided one makes apriori an assumption that h_- has only a single bound state (namely, its ground state) regardless of whether supersymmetry is broken or not.

To this end we consider (22), in which R_- may be replaced by $R_P = \langle x|\,\mathcal{P}\frac{1}{h_- - \omega^2}\,|x\rangle$ as mentioned in the discussion preceding (22). Since h_- is assumed to have a single bound state, R_P contains only scattering states of h_-, whose corresponding continuous eigenvalues $E = k^2 + m^2$ start at $E = m^2$. We may therefore write the spectral resolution of R_P as

$$R_P = \int\limits_{m^2}^{\infty} dE \frac{|\psi_E(x)|^2}{E - \omega^2} = \int\limits_{-\infty}^{\infty} dk \frac{\rho_k(x)}{k^2 + m^2 - \omega^2} \tag{36}$$

where $\rho_k(x) = 2\pi|k||\psi_k(x)|^2$. In terms of ρ_k the extremum condition (22) becomes

$$\frac{d}{dx}\frac{\rho_k(x)}{\Psi_0^2(x)} = \frac{d}{dx}\frac{1}{\Psi_0^2(x)} \tag{37}$$

whose general solution is

$$\rho_k(x) = 1 + c_k \Psi_0^2(x) \tag{38}$$

where c_k is an integration constant. Since by definition $\rho_k(x)$ cannot blow up at infinity, we must set $c_k = 0$ in the case of broken supersymmetry. Whether supersymmetry is broken or not $\rho_k(x)$ obviously obtains the asymptotic value of 1 as $x \to \pm\infty$. Therefore, the scattering states $\psi_k(x)$ of h_- are given by

$$\psi_k(x) = \sqrt{1 + c_k \Psi_0^2(x)}\; e^{i\alpha_k(x)} \tag{39}$$

where α_k is a real phase. Substituting these functions into the eigenvalue equation for h_- and considering its asymptotic behaviour as $x \to \pm\infty$ we see, using the boundary condition $g\sigma(\pm\infty) = \pm m$ that the phase becomes that of a free particle, which is obvious, but unimodularity of the phase factor implies further that there be only right moving or only left moving wave in $\psi_k(x)$. Therefore, h_- must be reflectionless.

The physical significance of the CCGZ kinks is as follows: As was mentioned in the introduction, the dynamical properties of (1) are consequences of the fact that the ("large N") effective potential $V_{\text{eff}}(\sigma)$ extracted from (4) has two symmetric equivalent minima at $\langle\sigma\rangle_{\text{vac}} = \pm\sigma_0 \neq 0$. This causes a dynamical breakdown of the discrete (Z_2) chiral symmetry of (2) under the transformation $\psi \to \gamma_5\psi$, $\sigma \to -\sigma$, where the fermions fluctuating near the $\langle\sigma\rangle_{\text{vac}} = \pm\sigma_0$ ground state acquire dynamical mass $m = \pm g\sigma_0$[1]. In a similar manner to the appearance of kinks in classical field theories with potentials exhibiting spontaneous symmetry breaking, one should expect similar configurations to appear in field theories whose *effective* potential implies dynamical symmetry breaking. The CCGZ kinks (anti-kinks) are precisely such static space-dependent $\sigma(x)$ configurations that interpolate between the two minima of $V_{\text{eff}}(\sigma)$, and our calculations provide an explicit proof that they are indeed extremal points of (4). The various states appearing in the background of the CCGZ kink may be deduced by calculating the "partition function" of the Dirac field fluctuations in that specific $\sigma(x)$ configuration for a finite time lapse T (i.e.-the trace over the time evolution operator e^{-iHT}) . This has been done explicitly in [2]. The result is

$$\mathrm{Tr}\, e^{-iHT} = \sum_{n=0}^{2N} \frac{(2N)!}{(n)!\,(2N-n)!}\, exp\left[-\frac{i}{2}\int_0^T dt \int_{-\infty}^{\infty} dx \left(\sigma_{kink}^2 - \sigma_0^2\right)\right] \cdot$$
$$exp\left\{iNT\left(\sum_i \omega_i\left[\sigma_{kink}\right] - \sum_i \omega_i\left[\sigma_0\right]\right) - in\omega_b\left[\sigma_{kink}\right]T\right\} . \qquad (40)$$

In this equation n is the total number of fermions and anti-fermions that are trapped in the single bound state of the kink, $\omega_i[\sigma] = \sqrt{E_i}$ is the energy of the $i-th$ state in the background of σ[19] and ω_b is the energy of the bound state, which is zero for the CCGZ kinks. Using this and also the fact that in this background $h_\pm$ in (29) are isospectral, we see that all terms in the second exponent in (40) are cancelled, leaving only the first exponent which is the mass of the kink, namely[20],

$$M_{kink} = \frac{1}{2}\int_{-\infty}^{\infty}\left[\sigma_0^2 - \sigma_{kink}^2\right] = \frac{mN}{Ng^2}\,. \tag{41}$$

Therefore, we see that all states contributing to (40) are degenarate in energy (all having the kink mass as energy) which is a direct result of the fact that $\omega_b = 0$. Clearly there are 2^{2N} states in all, that form a huge reducible supermultiplet of $\mathcal{O}(2N)$. Its decomposition into irreducible components is clear from the combinatorial prefactors in (40) that simply tell us that the various bound states in the kink fall into anti-symmetric tensor representations of $\mathcal{O}(2N)$ (where the integer n is the rank of the tensor)[2].

The conclusion that in the sector of unbroken supersymmetry the only possible extremal static $\sigma(x)$ configurations are simply either CCGZ kinks or anti-kinks implies that many bound states of kinks and anti-kinks could not form. As an example of such an unstable assembly we consider a $\sigma(x)$ configuration made of n kinks and $n-1$ anti-kinks which are interlaced alternately. This configuration has obviously the correct asymptotics since it interpolates between the two vacua at the two ends of the world. It leads to a manifestly normalisable Ψ_0 function with a zero energy eigenvalue, hence it respects supersymmetry, but it *does not* satisfy the extremum condition and thus, cannot be stable. Thus, for example, such multi-kink configurations must be excluded from the discussion of finite temperature properties of the Gross-Neveu model in [7], but this would probably cause no qualitative change in the picture presented there.

As was discussed above, CCGZ kinks which are the only possible static extrema of (4) that do not break supersymmetry lead to an h_- operator whose potential $g^2\sigma^2 - g\sigma'$ is reflectionless and supports only a single bound state, which is therefore the ground state of h_-. Since supersymmetry is unbroken its binding energy must vanish, which implies in turn, that fluctuations of fermions trapped in this potential do not react back on the $\sigma(x)$ field [2]. Therefore, obvious conceivable generalisations of the CCGZ kinks, namely, configurations of the form $\sigma(x) = m\,tanh[\frac{m(x-x_0)}{n}]$; $n \in \mathbf{Z}$ which interpolate between the two vacua and lead to a reflectionless h_- operators binding n states[6] are not extremal points of (4). Back-reaction of fermionic states they trap probably destabilise them.

We close this section by checking explicitly that the CCGZ kink configurations obtained above indeed extremise the effective action in (4). Substituting the kink configuration in (26) and the explicit expressions (28) and (21) for R_- and R_+ into the extremum condition (8) we find that the pole at $\omega^2 = 0$ disappears from the right hand side of (8) in accordance with (18) leaving in the sum over frequencies only contributions from the scattering states. Thus, Eq.(8) becomes

$$\left[1 + iNg^2 \int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi} \frac{1}{\sqrt{m^2 - \omega^2 - i\epsilon}}\right] g\sigma(x) = 0, \tag{42}$$

implying that the term in the square brackets on the left hand side must vanish. But vanishing of the latter is precisely the Minkowsky space gap-equation of the Gross-Neveu model (5) for the dynamical mass m and it must therefore hold, confirming that the kinks in (26) are indeed solutions of the extremum condition (8).

IV Conclusion

In this paper we have solved the extremum condition on the effective action of the two dimensional Gross-Neveu model for *static* $\sigma(x)$ configurations in the large N limit. Our method of calculation was direct and straightforward, making use of elementary properties of one dimensional Schrödinger operators. Our explicit calculations serve as a clean and constructive proof that the CCGZ kink configurations are indeed extremal static $\sigma(x)$ configurations of the effective action. Unfortunately, our method of calculation is blind to the other extremal static $\sigma(x)$ configurations of the Gross-Neveu model reported in [2]. The reason for this drawback must be related to the fact that the only physical scale that appeared explicitly in our calculation was the dynamically generated mass m which is also the natural scale of the CCGZ kinks. In order to see the other static configurations of [2] one must some how incorporate their mass scale into the extremum condition. Our method may be applied also to a host of other two dimensional field theories, and in particular, to field theories that *do not* involve reflectionless static configurations, where inverse scattering methods are useless.

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